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The non-static contribution to the effective potential for the t–J model

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Abstract. The leading non-static contribution to the effective potential in a RVB (resonating valence bond) phase for the t–J model is characterized in terms of a canonical integral. It is shown that a Chern–Simons term does not arise from such an integral at low doping; however, at higher doping this integral suggests that such a term could appear.

1. Introduction

In theoretical modelling of high-temperature superconductivity it is widely recognised that the coupling of charge and spin degrees of freedom in CuO_2 planes of the perovskite superconductors is an essential ingredient. Anderson [1] was the first to suggest that strongly correlated Hubbard [2] models with spin $\frac{1}{2}$ would be an appropriate starting point for investigations. Later, Zhang and Rice [3] and Jefferson *et al* [4] gave arguments for the validity of such one-band models starting from a more general model involving relevant d orbitals for the Cu and p orbitals for the O. In the strong correlation limit, the appropriate form of the Hubbard model becomes the t–J model, for which the Hamiltonian can be written as

$$H_{t-J} = -t\sum_{\langle ij\rangle} P_{0,1}(C_{i\sigma}^{+}C_{j\sigma} + C_{j\sigma}^{+}C_{i\sigma})P_{0,1} + J\sum_{\langle ij\rangle} P_1(\boldsymbol{S}_i\cdot\boldsymbol{S}_j - \tfrac{1}{4})P_1. \tag{1}$$

The operator P_1 projects onto states with single occupancy at each site and $P_{0,1}$ projects out double occupancy. $C_{i\sigma}^{+}$ $(C_{i\sigma})$ represents a creation (annihilation) operator at site i for a carrier with spin σ. As usual $\langle ij\rangle$ denotes that i and j are nearest neighbour sites. In the limit of single carrier occupation per site the Hamiltonian reduces to the Heisenberg model.

In two-dimensional space the statistics of quantum mechanical particles may be anomalous i.e. neither fermionic nor bosonic. This implies parity and time reversal violating states. There have recently been investigations of the consequences of continuous field theories with so called Chern–Simons terms which can give rise to such states [5]. Generally, given a system with a symmetry described by a current J_μ, the Chern–Simons term [5] in a Lagrangian formulation is given by the following contribution to the Lagrangian density:

$$\mathcal{L}_{\text{Chern-Simons}} = -(1/8\alpha)\epsilon_{\mu\nu\lambda}A^{\mu}F^{\nu\lambda} \tag{2}$$

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where

$$J_\mu = (1/4\alpha)\epsilon_{\mu\nu\lambda}F^{\nu\lambda} \tag{3}$$

$$F_{\nu\lambda} = \partial_\nu A_\lambda - \partial_\lambda A_\nu. \tag{4}$$

Here the three co-ordinates are the two spatial dimensions and the temperature (which is an imaginary time variable); $\exp(i\alpha)$ is the phase change of a wavefunction when two quasi-particles are exchanged. Whereas the consequences of such a term in continuum theory have been to a certain extent examined, the question of its existence, starting from a microscopic lattice Hamiltonian such as the t–J model, has received little attention. One of the few discussions has been due to Aitchison and Mavromatos [6] who have argued on symmetry grounds that such a term could arise in a mean field version of the t–J model. However the question cannot be settled on purely symmetry grounds and has to be a consequence of many-body effects. Moreover the mean field version of the t–J model misses out the important strong correlation effects which should be the key to the existence of anomalous statistics. The experimental situation has not led to a clear cut resolution since there is disagreement on the interpretation of experiments [7] (such as those involving light scattering). Since we do not yet have a rigorous theory for such effects it is essential to apply as many different approaches as possible and come to a consensus. We will calculate the partition function for the t–J model using a functional integral representation, Hubbard–Stratonovich transformation in the order parameter and a form of high-temperature expansion. The effect of the projection operators is directly taken into account by restriction of states in performing the trace in the partition function. This formulation [8] has already been used to calculate the Landau–Ginzburg functional in the leading order (or static approximation) of the high-temperature expansion. However, as should be clear from (2), the topological term involves derivatives and so changes with respect to the temperature variable. Consequently a static approximation in the high-temperature formulation will be inadequate.

2. The partition function

A sum over any complete set of states is sufficient to calculate the partition function Z where

$$Z = \sum_{\text{states}} \exp(-\beta H_{t-J}). \tag{5}$$

Owing to the projection operators in H_{t-J} the sum can be restricted to the subspace of no double occupancy. In order to have a hope of finding a Chern–Simons term we need to be in a T and P violating phase. Anderson [1] has suggested that an order parameter associated with such phases is given by the expectation value Δ_{ij} of

$$b_{ij} = (1/\sqrt{2})(C_{i\uparrow}C_{j\downarrow} - C_{i\downarrow}C_{j\uparrow}). \tag{6}$$

One such phase is given by a Δ_{ij} which is translationally invariant and has values Δ_x and Δ_y along the links in the x and y directions respectively. It has been shown [8] that there is a static saddle point solution of the functional integral with $\Delta_y = \Delta_x \exp(\mathrm{i}\phi)$ (where $\phi \in [\pi/2, \pi]$) for sufficiently low doping and temperature. Now it has been shown [7] that there is a static saddle point solution of the functional integral favouring such a phase for sufficiently low doping and temperature. We will calculate the leading order 'time' dependent contribution to the functional integral for the partition function in this phase. As will be clear presently, we can write

$$Z = \int \prod_{\tau,\langle i,j\rangle} \mathrm{d}\Delta^*_{ij}(\tau)\,\mathrm{d}\Delta_{ij}(\tau)\, \exp\left(-\int_0^\beta \mathrm{d}\tau\, \mathcal{L}[\Delta(\tau)]\right). \tag{7}$$

We have continuous 'time' but a discrete spatial lattice. The Lagrangian like object $\mathcal{L}$ is, in this context, known as a Landau–Ginzburg functional or effective potential. In order to set up the functional integral formulation of the partition function it is possible to follow the methods [9] applied to the Anderson impurity problem even though in that case there is no projection operator controlling the occupation of sites. Now the Heisenberg terms can be written as

$$S_i \cdot S_j = \tfrac{1}{4} - \tfrac{1}{2} C^+_{i\sigma} C_{j\sigma} C^+_{j\sigma'} C_{i\sigma'}. \tag{8}$$

This can be further simplified by noting that

$$\tfrac{1}{2} C^+_{i\sigma} C_{j\sigma} C^+_{j\sigma'} C_{i\sigma'} = b^+_{ij} b_{ij}. \tag{9}$$

It is standard at this juncture [5,8] to use the Hubbard–Stratonovich identity

$$\int_{-\infty}^{\infty} \mathrm{d}\Delta\, \exp[-(\Delta - u)^2 + v^2] = \sqrt{\pi}\exp(v^2) \tag{10}$$

where u and v can be operators provided that they commute with each other. This leads to [8]

$$\begin{aligned} Z = \int \prod_{\langle i,j\rangle} \left(D^2\Delta_{ij} \frac{\beta}{\pi J}\right) \mathrm{Tr}\Bigg[P_{0,1} \exp[-\beta(H_t + H_0)] \\ \times T\exp\Bigg(-\sum_{\langle i,j\rangle} \int_0^\beta \mathrm{d}\tau\, [(1/J)|\Delta_{ij}(\tau)|^2 \\ + \Delta_{ij}(\tau) b^+_{ij}(\tau) + \Delta^*_{ij}(\tau) b_{ij}(\tau)]\Bigg)\Bigg] \end{aligned} \tag{11}$$

where we have used the standard representation of the time evolution operator in the interaction representation, and

$$b_{ij}(\tau) = \exp[\tau(H_0 + H_t)] b_{ij}(0) \exp[-\tau(H_0 + H_t)] \tag{12a}$$

$$b^+_{ij}(\tau) = \exp[\tau(H_0 + H_t)] b^+_{ij}(0) \exp[-\tau(H_0 + H_t)]. \tag{12b}$$

H_0 is defined by [8]

$$H_0 = (\mu_{\mathrm{h}} - J) \sum_i (n_i - 1) \tag{13}$$

μ_h being the hole chemical potential. H_t is the hopping term (proportional to t) in H_{t-J}. $\Delta_{ij}(\tau)$ is a complex function and

$$D^2\Delta_{ij} = \mathrm{d}\Delta_{ij}^{(x)}\,\mathrm{d}\Delta_{ij}^{(y)} \tag{14}$$

with $\Delta_{ij} = \Delta_{ij}^{(x)} + \mathrm{i}\Delta_{ij}^{(y)}$. For each link there corresponds only one Δ_{ij}. In order to introduce a gauge field representation for Δ_{ij} it will be useful to choose a direction convention for links. For our two-dimensional lattice with lattice points labelled by (n, m) we can, for example, adopt the following convention for directions:

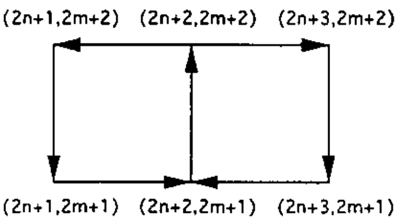

Figure 1. Convention for directional links.

If we follow the standard *ansatz* [10] that

$$\Delta_{ij}(\tau) \simeq \Delta_0(\beta)\exp(\mathrm{i}\theta_{ij}(\tau)) \tag{15}$$

where the magnitude $\Delta_0(\beta)$ is independent of τ, then the gauge field [9] appears on writing

$$\begin{aligned}
\theta_{r,r+a\hat{x}} &= 2\phi(r) + gaA_x(r) \\
\theta_{r,r+a\hat{y}} &= 2\phi(r + a\hat{y}) - gaA_y(r) \\
\theta_{r,r-a\hat{x}} &= 2\phi(r) - gaA_x(r - a\hat{x}) \\
\theta_{r,r-a\hat{y}} &= 2\phi(r - a\hat{y}) + gaA_y(r - a\hat{y}).
\end{aligned} \tag{16}$$

Here r is a lattice site, $\hat{x}$ and $\hat{y}$ are unit vectors in the x and y directions and a is a lattice spacing. The direction of the link in θ_{ij} will go from i to j. The functions $A_x(r)$ and $A_y(r)$ are gauge fields. An important aspect of the functional representation is the periodicity [9] of Δ_{ij},

$$\Delta_{ij}(\tau + \beta) = \Delta_{ij}(\tau) \tag{17}$$

and so it can be expanded as a Fourier series,

$$\Delta_{ij}(\tau) = \sum_{\nu=-\infty}^{\infty} \tilde{\Delta}_{ij,\nu}\exp(-\mathrm{i}\Omega_\nu\tau) \tag{18}$$

where

$$\Omega_\nu = 2\pi\nu/\beta.$$

It should now be clear as to the meaning of static and non-static contributions. In the static approximation,

$$\Delta_{ij}(\tau) \simeq \tilde{\Delta}_{ij,0} \tag{19}$$

and so the leading (mildest) form of non-static contribution $\Delta_{ij}^{(1)}(\tau)$ is given by

$$\Delta_{ij}^{(1)}(\tau) = \tilde{\Delta}_{ij,1}\exp(-\mathrm{i}\Omega_1\tau) + \tilde{\Delta}_{ij,-1}\exp(\mathrm{i}\Omega_1\tau). \tag{20}$$

(This form of non-static contribution is more appropriate than a Taylor expansion which does not respect periodicity in time.) In (16) only $A_1(= A_x)$ and $A_2(= A_y)$ components have been introduced. Consequently the time component of A does not occur and we can regard this as the manifestation of choosing the gauge $A_0 = 0$. In this gauge the Chern–Simons term reduces to

$$\mathcal{L}_{\text{Chern-Simons}} = -(1/4\theta)[A_2(x)\partial^0 A_1(x) - A_1(x)\partial^0 A_2(x)]. \tag{21}$$

This can emerge from a term

$$\Delta_{ij}(\tau)\partial^0\Delta_{jk}^*(\tau)$$

when

$$\theta_{jk} = \theta_{r,r+a\hat{x}} \tag{22a}$$

$$\theta_{ij} = \theta_{r+a\hat{y},r}. \tag{22b}$$

Using (15) we can deduce that

$$\Delta_{ij}(\tau)\partial^0\Delta_{jk}^*(\tau) \simeq |\Delta_0(\beta)|^2[1 + \mathrm{i}(\theta_{ij}(\tau) - \theta_{jk}(\tau))][-\mathrm{i}\dot{\theta}_{jk}(\tau)]. \tag{23}$$

We consider the piece of this expression

$$g^2a^2(A_y(r) - A_x(r))\dot{A}_x(r) = g^2a^2\{A_y(r)\dot{A}_x(r) - \tfrac{1}{2}(\partial/\partial\tau)[A_x(r)]^2\}. \tag{24}$$

Since in the expression for the partition function we are actually interested in

$$\int_0^\beta \mathrm{d}\tau\, \Delta_{ij}(\tau)\dot{\Delta}_{jk}^*(\tau)$$

the total derivative in (24) does not contribute owing to the periodicity of $A_x(r)$ as a function of τ. Similarly the term $\dot{\Delta}_{ij}(\tau)\Delta_{jk}^*(\tau)$ gives rise to $-g^2a^2A_x(r)\dot{A}_y(r)$ and so the term $\Delta\mathcal{L}$

$$\Delta\mathcal{L} = \int_0^\beta \mathrm{d}\tau\,(\Delta_{ij}(\tau)\dot{\Delta}_{jk}^*(\tau) - \dot{\Delta}_{ij}(\tau)\Delta_{jk}^*(\tau)) + \mathrm{CC} \tag{25}$$

contains the Chern–Simons contribution. The hope that a Chern–Simons term may be generated from fluctuations in the theory in the presence of holes has some indirect support from a calculation in field theory. A fermionic isospin doublet Ψ, when coupled by a Yukawa term to a c-number isospin vector field, generates an expectation value of the fermionic current. This expectation value is proportional to the topological current [11] when the c-number isospin vector field is a soliton solution of the non-linear σ-model. The non-linear σ-model is the continuum version of the Heisenberg model. By a series of arguments [12] the Chern–Simons term can be related to a non-local bilinear interaction of the topological current.

3. The non-static approximation

It is possible to adopt the method for obtaining the non-static approximation in the Anderson model used by Morandi *et al* [9] for the case that the Hilbert space is not constrained by projection operators. In (11) we write

$$\Delta_{ij}(\tau) = \Delta_{ij,0} + \tilde{\Delta}_{ij}(\tau) \tag{26}$$

then

$$Z = \int \prod_{(ij)} D^2\Delta_{ij,0}\frac{\beta}{\pi J} D^2\tilde{\Delta}_{ij}(\tau) \exp\left(-\sum_{(ij)} \int_0^\beta \mathrm{d}\tau\, \frac{1}{J} |\Delta_{ij}(\tau)|^2 \right) \times \mathrm{Tr}\left[P_{0,1} \exp[-\beta(H_0 + H_t)] T \exp\left(-\sum_{(ij)} \int_0^\beta \mathrm{d}\tau\, (\Delta_{ij,0} b^+_{ij}(\tau) + \Delta^*_{ij,0} b_{ij}(\tau) + \tilde{\Delta}_{ij}(\tau) b^+_{ij}(\tau) + \tilde{\Delta}^*_{ij}(\tau) b_{ij}(\tau)) \right) \right]. \tag{27}$$

We can now change the interaction to one with respect to the Hamiltonian H'

$$H' = H_0 + H_t + \sum_{(ij)} [\Delta_{ij,0} b^+_{ij}(0) + \Delta^*_{ij,0} b_{ij}(0)] \tag{28}$$

and then

$$Z = \int \prod_{(ij)} \left(D^2\tilde{\Delta}_{ij}(\tau) \frac{\beta}{\pi J} \right) D^2\Delta_{ij,0} \exp\left(-\sum_{(ij)} \int_0^\beta \mathrm{d}\tau\, \frac{1}{J} |\Delta_{ij}(\tau)|^2 \right) \times \mathrm{Tr}\left\{ P_{0+1} \exp\left[-\beta\left(H_0 + H_t + \sum_{(ij)} [\Delta_{ij,0} b^+_{ij}(0) + \Delta^*_{ij,0} b_{ij}(0)] \right) \right] \times T \exp\left(-\sum_{(ij)} \int_0^\beta \mathrm{d}\tau\, [\tilde{\Delta}_{ij}(\tau) \tilde{b}^+_{ij}(\tau) + \tilde{\Delta}^*_{ij}(\tau) \tilde{b}_{ij}(\tau)] \right) \right\} \tag{29}$$

where

$$\tilde{b}_{ij}(\tau) = \exp\left[\tau\left(H_0 + H_t + \sum_{(ij)} [\Delta_{ij}(0) b^+_{ij}(0) + \Delta^*_{ij}(0) b_{ij}(0)] \right) \right] \times b_{ij}(0) \exp\left[-\tau\left(H_0 + H_t + \sum_{(ij)} [\Delta_{ij}(0) b^+_{ij}(0) + \Delta^*_{ij}(0) b_{ij}(0)] \right) \right] \tag{30}$$

(and $\tilde{b}^+_{ij}(\tau)$ is defined in a similar way). (28) follows from the general fact that

$$\exp(-\mathcal{H}\tau) = \exp(-\mathcal{H}_0\tau) T \exp\left(-\int_0^\tau (\mathcal{H} - \mathcal{H}_0)\, \mathrm{d}\tau' \right) \tag{31}$$

for the interaction representation with respect to $\mathcal{H}_0$. This reformulation extracts rather clearly the non-static contribution. We can perform the trace in (28) up to second order $\tilde{\Delta}_{ij}$ and this will suffice to calculate the non-static corrections to $\mathcal{L}$ in lowest order. If our method of calculation is valid to sufficiently low temperature (i.e. there is no phase transition as we lower T), then we will be able to provide evidence for or against the existence of a Chern–Simons term. The value of $\Delta_{ij,0}$ will determine the nature of the phase about which we are finding excitations, and has already been calculated [8]. To second order in $\tilde{\Delta}$,

$$
\begin{aligned}
Z \simeq \int \prod_{(ij)} \mathrm{d}\Delta^{(x)}_{ij,0}\,\mathrm{d}\Delta^{(y)}_{ij,0} \left(\frac{\beta}{\pi J}\right)^2 4\,\mathrm{d}^2\tilde{\Delta}^{(x)}_{ij,1}\,\mathrm{d}^2\tilde{\Delta}^{(y)}_{ij,1} \left(\frac{\beta}{\pi J}\right)^4 \\
\times \exp\Big(- \sum_{(ij)} \frac{\beta}{J}[(\Delta^{(x)}_{ij,0})^2 + (\Delta^{(y)}_{ij,0})^2 + 2|\tilde{\Delta}^{(x)}_{ij,1}|^2 + 2|\tilde{\Delta}^{(y)}_{ij,1}|^2]\Big) \\
\times \operatorname{Tr}\Big[P_{0,1} \exp(-\beta\mathcal{H}') \\
\times \Big(1 - \frac{1}{2}\sum_{(i,j)}\sum_{(i',j')} \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'\, T\{[\tilde{\Delta}_{ij,1}(\tau)\tilde{b}^{+}_{ij}(\tau) \\
+ \tilde{\Delta}^{*}_{ij,1}(\tau)\tilde{b}_{ij}(\tau)][\tilde{\Delta}_{i'j',1}(\tau')\tilde{b}^{+}_{i'j'}(\tau') + \tilde{\Delta}^{*}_{i'j',1}(\tau')\tilde{b}_{i'j'}(\tau')]\}\Big)\Big]
\end{aligned}
\tag{32}
$$

and so equivalently we have

$$
\begin{aligned}
\int_0^\beta \mathrm{d}\tau\, \mathcal{L}[\Delta(\tau)] \approx \sum_{(i,j)} \frac{\beta}{J}[(\Delta^{(x)}_{ij,0})^2 + (\Delta^{(y)}_{ij,0})^2 + 2|\tilde{\Delta}^{(x)}_{ij,1}|^2 + 2|\tilde{\Delta}^{(y)}_{ij,1}|^2 \\
- \log \operatorname{Tr}\Big[P_{0,1} \exp(-\beta H') \\
\times \Big(1 - \frac{1}{2}\sum_{(i,j)}\sum_{(i',j')} \int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'\, T\{[\tilde{\Delta}_{ij,1}(\tau)\tilde{b}^{+}_{ij}(\tau) \\
+ \tilde{\Delta}^{*}_{ij,1}(\tau)\tilde{b}_{ij}(\tau)][\tilde{\Delta}_{i'j',1}(\tau')\tilde{b}^{+}_{i'j'}(\tau') + \tilde{\Delta}^{*}_{i'j',1}(\tau')\tilde{b}_{i'j'}(\tau')]\}\Big)\Big].
\end{aligned}
\tag{33}
$$

The evaluation of this trace is the major task facing us. It will be convenient to replace the term providing time ordering in (33) by the equivalent expression

$$
\begin{aligned}
\frac{1}{2}\int_0^\beta \mathrm{d}\tau \int_0^\beta \mathrm{d}\tau'\, T(\ldots) = \int_0^\beta \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \sum_{(i,j)}\sum_{(i',j')} [\tilde{\Delta}_{ij,1}(\tau)\tilde{\Delta}^{*}_{i'j',1}(\tau')\tilde{b}^{+}_{ij}(\tau)\tilde{b}_{i'j'}(\tau') \\
+ \tilde{\Delta}^{*}_{ij,1}(\tau)\tilde{\Delta}_{i'j',1}(\tau')\tilde{b}_{ij}(\tau)\tilde{b}^{+}_{i'j'}(\tau')].
\end{aligned}
\tag{34}
$$

We can visualize through diagrams the contributions to Ω, where

$$
\begin{aligned}
\Omega = \operatorname{Tr}\Big(P_{0,1}\exp(-\beta\mathcal{H}') \int_0^\beta \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' [\tilde{\Delta}_{ij,1}(\tau)\tilde{\Delta}^{*}_{i'j',1}(\tau')\tilde{b}^{+}_{ij}(\tau)\tilde{b}_{i'j'}(\tau') \\
+ \tilde{\Delta}^{*}_{ij,1}(\tau)\tilde{\Delta}_{i'j',1}(\tau')\tilde{b}_{ij}(\tau)\tilde{b}^{+}_{i'j'}(\tau')]\Big).
\end{aligned}
\tag{35}
$$

The individual terms for which we need to calculate the trace can be expressed in terms of the following operator elements:

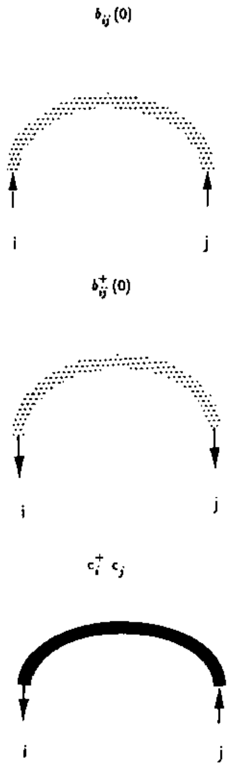

Figure 2. Basic diagrams.

Let us examine how diagrams arise. On using (30),

$$\Omega = \int_0^{\beta} \mathrm{d}\tau \int_0^{\tau} \mathrm{d}\tau' [\![\mathrm{Tr}\{\exp[-(\beta-\tau)H']\tilde{\Delta}_{ij,1}(\tau)\tilde{\Delta}^{*}_{i'j',1}(\tau')b^{+}_{ij}(0)$$

$$\times \exp[-(\tau-\tau')H']b_{i'j'}(0)\exp(-\tau' H')\} + (\mathrm{cc\ with\ } (i,j) \leftrightarrow (i'j'))]\!]. \tag{36}$$

The terms that contribute to Ω are proportional to

(i) $\tilde{\Delta}_{ij,1}(\tau)\tilde{\Delta}^{*}_{i'j',1}(\tau')$

(ii) $\Delta_{lm,0}\Delta^{*}_{l'm',0}\tilde{\Delta}_{ij,1}(\tau)\tilde{\Delta}_{i'j',1}(\tau')$

(since we are calculating a trace, an equal number of creation and annihilation operators must act on each state at any site). For local contributions the lattice sites involved need to be continuous. The terms giving (i) can be summarized by

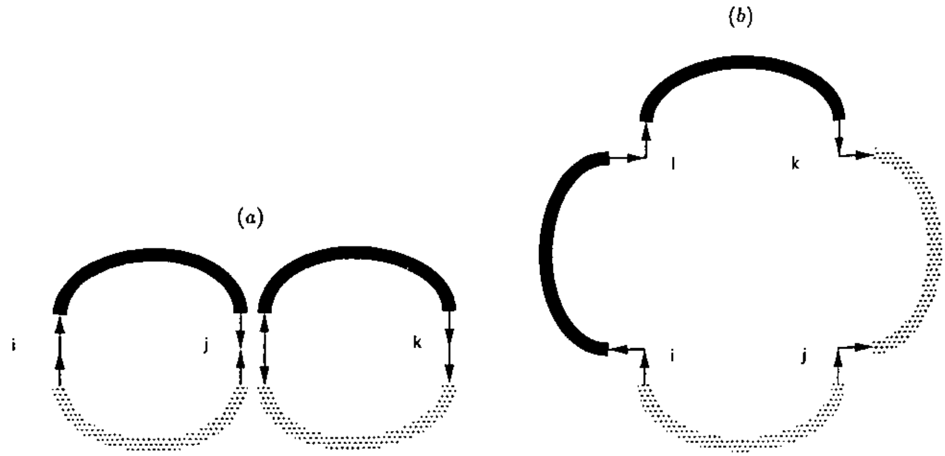

Figure 3.

Similarly (ii) is derived from

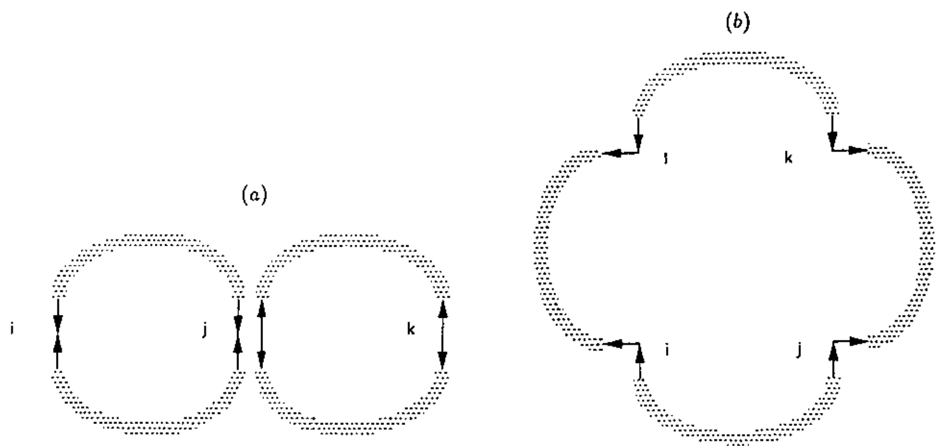

Figure 4.

The expression corresponding to figure 3(a) (I_{3a} say) is

$$I_{3a} = \sum_w \sum_v \sum_u \underset{k}{\langle} w| \underset{j}{\langle} v| \underset{i}{\langle} u| \exp[-(\beta - \tau) H_0]$$
$$\times \exp[-(\beta - \tau) H_t] \tilde{\Delta}_{ij,1}(\tau) \tilde{\Delta}^*_{jk,1}(\tau') b^+_{ij}(0)$$
$$\times \exp[-(\tau - \tau')(H_0 + H_t)] b_{jk}(0) \exp[-\tau'(H_0 + H_t)] |u\rangle_i |v\rangle_j |w\rangle_k \tag{37}$$

where the summation variables can take the values of 0 (empty site), ↑ (single up spin occupation) and ↓ (single down spin occupation). Similar expressions can be associated with figures 3(b), 4(a) and 4(b). Although the computations are, in principle, straightforward, there are intermediate steps of some algebraic complexity.

Consequently, as a check, we have found it advantageous to use REDUCE, a computer program designed for symbolic manipulation. A product of a string of creation and annihilation operators can thus readily be made to normal order which is a manipulation at the heart of evaluating matrix elements such as those in (37). The expression for the partition function is expanded in β, the magnitude of the order parameter (assumed small), t and the hole fugacity [7] which vanishes at half-filling.

4. The non-static effective potential

We will simplify $\Delta\mathcal{L}$ prior to checking whether such a form appears in our calculation of the non-static effective potential. Owing to periodicity, only the constant part of the integral in (25) contributes, and gives

$$\Delta\mathcal{L} = \frac{2\pi\mathrm{i}}{\beta}(-\tilde{\Delta}_{ij,-1}\tilde{\Delta}^{*}_{jk,-1} + \tilde{\Delta}_{ij,1}\tilde{\Delta}^{*}_{jk,1}) + \mathrm{cc}. \tag{38}$$

In the calculations of the contributions of figures 3(*a*), 3(*b*), 4(*a*) and 4(*b*) we find that the results can be written in terms of the canonical integral

$$I_{ijk}(a,c) = \int_0^\beta \mathrm{d}\tau \int_0^\tau \mathrm{d}\tau' \tilde{\mathcal{D}}_{ij,1}(\tau)\tilde{\mathcal{D}}^{*}_{jk,1}(\tau') \exp(a\tau)\exp(c\tau') \tag{39}$$

where

$$\tilde{\mathcal{D}}_{ij,1}(\tau) = \tilde{\Delta}_{ij,1}\exp[-(2\pi\mathrm{i}/\beta)\tau] + \tilde{\Delta}_{ij,-1}\exp[(2\pi\mathrm{i}/\beta)\tau]. \tag{40}$$

A typical contribution to Ω has the form

$$\Delta^{*}_{ij,0}\Delta_{jk,0}[I^{(2,0)}_{ijk}(2\bar{\mu}_{\mathrm{h}}, -2\bar{\mu}_{\mathrm{h}}) + I^{(0,2)}_{ijk}(2\bar{\mu}_{\mathrm{h}}, -2\bar{\mu}_{\mathrm{h}}) - 2I^{(1,1)}_{ijk}(2\bar{\mu}_{\mathrm{h}}, -2\bar{\mu}_{\mathrm{h}})]$$

and so we will consider the canonical integral $I^{(n,m)}_{ijk}(-c,c)$. ($\beta\bar{\mu}_{\mathrm{h}}$ is $\log z$ and z is the hole fugacity.) We consider the limits

(i) $|\beta c| \ll 2\pi$

or

(ii) $|\beta c| \gg 2\pi$.

Since $c = -p\bar{\mu}_{\mathrm{h}}$ where p is a positive integral these limits are equivalent to

(i)′ $|\beta\bar{\mu}_{\mathrm{h}}| \ll 2\pi$

or

(ii)′ $|\beta\bar{\mu}_{\mathrm{h}}| \gg 2\pi$.

For case (i) we find

$$I^{(0,0)}_{ijk}(-c,c) \simeq \frac{-\mathrm{i}}{2\pi}\beta^3(\tilde{\Delta}_{ij,1}\tilde{\Delta}^*_{jk,1} - \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,-1}) \tag{41a}$$

$$I^{(0,2)}_{ijk}(-c,c) \simeq \mathrm{i}\beta^4(1/4\pi^3 - 1/6\pi)(\tilde{\Delta}_{ij,1}\tilde{\Delta}^*_{jk,1} - \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,-1}) + \cdots \tag{41b}$$

$$I^{(2,0)}_{ijk}(-c,c) \simeq \mathrm{i}\beta^4(1/4\pi^3 - 1/6\pi)(\tilde{\Delta}_{ij,1}\tilde{\Delta}^*_{jk,1} - \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,-1}) + \cdots \tag{41c}$$

$$I^{(1,1)}_{ijk}(-c,c) \simeq -\mathrm{i}\beta^4(1/8\pi^3 + 1/6\pi)(\tilde{\Delta}_{ij,1}\tilde{\Delta}^*_{jk,1} - \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,-1}) + \cdots \tag{41d}$$

and there are similar expressions for $I^{(n,m)}_{ijk}(-c,c)$ with other n and m. The important point to note is the appearance of the combination $(\tilde{\Delta}_{ij,1}\tilde{\Delta}^*_{jk,1} - \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,-1})$ which is precisely that which occurs in $\Delta\mathcal{L}$, the Chern–Simons term. For case (ii) this combination does not appear in $I^{(n,m)}_{ijk}(-c,c)$, e.g.

$$\begin{aligned} I^{(0,2)}_{ijk}(-c,c) \simeq{}& (\tilde{\Delta}_{ij,1}\tilde{\Delta}^*_{jk,1} + \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,-1})(\beta^4/3\beta c) \\ &+ (\tilde{\Delta}_{ij,1}\tilde{\Delta}_{jk,1} + \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,1})(\beta^4/8\pi^2\beta c) \\ &+ (\tilde{\Delta}_{ij,1}\tilde{\Delta}^*_{jk,-1} - \tilde{\Delta}_{ij,-1}\tilde{\Delta}^*_{jk,1})(\mathrm{i}\beta^4/4\pi\beta c). \end{aligned} \tag{42}$$

Consequently for very low doping (which is case (ii)) this is an indication that the Chern–Simons term does not occur. Case (i) indicates that at higher doping such a term can exist. However our calculation is not refined enough to determine the transition between the two behaviours. The purpose of this calculation has been to show how qualitatively such a term can arise in a straightforward way in the t–J model. It will be important now to pursue this approach but to include other possible order parameters in the effective potential. It will then be possible to examine the detailed behaviour of the Chern–Simons term as a function of doping. In order to proceed with such calculations efficiently, we need to develop a systematic scheme. This is provided by the finite cluster method, details of which will be published elsewhere [13].

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